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Padé extension of Bergman-series solutions to nonlinear boundary-value problems in heat conduction

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Summary

Bergman-type series solutions involving iterated complementary error integrals are constructed for nonlinear boundary-value problems associated with heat conduction in a region bounded internally by a cylindrical or spherical surface. In particular, a small-time solution is developed when the nonlinear boundary condition is of the Stefan-Boltzmann type. This solution is extended via Padé approximants.

0. Introduction

Nonlinear boundary-value problems involving heat conduction in a half-space were recently treated by Tao [1] via series expansions in terms of iterated error integrals. Here, Bergman-type series solutions for analogous boundary-value problems but involving media bounded internally by cylindrical or spherical surfaces are obtained by an approach motivated by the Karal-Keller asymptotic wave-front expansion method earlier employed in inhomogeneous elastodynamics and viscoelastodynamics as well as in the dynamics of initially-stressed neo-Hookean materials [2–5].

In a recent paper by Barclay, Moodie and Tait [6], Padé approximants were used to extend the range of validity of wave-front expansions in inhomogeneous viscoelastic media of hereditary type. Here, a small-time Bergman series solution of a nonlinear boundary-value problem in heat conduction is extended via Padé-approximant methods.

1. The Bergman-type expansions

The nonlinear boundary-value problem considered here is specified by

$$\kappa\frac{\partial}{\partial r}\left[r^{2\nu+1}\frac{\partial u}{\partial r}\right]=\rho c r^{2\nu+1}\frac{\partial u}{\partial t},\quad r>a,\ t>0; \tag{1.1}$$

$$c_1\Phi(u(a,t))+c_2\frac{\partial u}{\partial r}(a,t)=\Psi(\eta),\quad t>0,\ \eta=t^{1/2}; \tag{1.2}$$

$$\lim_{t\to 0^+} u=U(r),\quad r>a; \tag{1.3}$$

$$u \text{ remains bounded as } r\to+\infty. \tag{1.4}$$

In the above, $u(r, t)$ represents the temperature distribution, κ the thermal conductivity of the medium, ρ its density and c the heat capacity per unit mass. The physical parameters κ, ρ and c are here assumed to be constant. The cases $\nu = 0$ and $\nu = +\frac{1}{2}$ give, in turn, boundary-value problems involving regions bounded internally by a cylindrical or spherical surface subject to a nonlinear condition.

It is assumed that the prescribed functions $\Phi(u(a, t))$ and $\psi(\eta)$ are analytic in their respective arguments so that, if we set $u_a \equiv u(a, t)$, then

$$\Phi(u_a) = \sum_{n=0}^{\infty} \phi_n u_a^n / n! \tag{1.5}$$

and

$$\Psi(\eta) = \sum_{n=0}^{\infty} \psi_n \eta^n / n! \tag{1.6}$$

where

$$\phi_n \equiv d^n \Phi / du_a^n \,|_{u_a=0}, \quad \Psi_n \equiv d^n \Psi / d\eta^n \,|_{\eta=0}. \tag{1.7,1.8}$$

The solution of the nonlinear boundary-value problem (1.1)–(1.4) is now sought in the form

$$u = T_0 + \sum_{n=0}^{\infty} a_n(r) \eta^n \mathrm{i}^n \operatorname{erfc}[R(r)\eta^{-1}] \tag{1.9}$$

where the iterated complementary error integral $\mathrm{i}^n \operatorname{erfc} \xi$ is defined recursively by

$$\mathrm{i}^n \operatorname{erfc} \xi = \int_{\xi}^{\infty} \mathrm{i}^{n-1} \operatorname{erfc} y \,\mathrm{d}y, \quad n \in \mathbb{Z}^+, \tag{1.10}$$

$$\mathrm{i}^0 \operatorname{erfc} \xi = \operatorname{erfc} \xi = 1 - \operatorname{erf} \xi = \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} \mathrm{e}^{-y^2} \,\mathrm{d}y, \tag{1.11}$$

and where the error integral $\operatorname{erf} \xi$ is given by

$$\operatorname{erf} \xi = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \mathrm{e}^{-y^2} \,\mathrm{d}y. \tag{1.12}$$

If we set

$$\Phi_n = \eta^n \mathrm{i}^n \operatorname{erfc}[R(r)\eta^{-1}], \tag{1.13}$$

then it is seen that

$$\frac{\partial \Phi_n}{\partial r} = -R' \eta^{n-1} \mathrm{i}^{n-1} \operatorname{erfc}[R(r)\eta^{-1}] = -R' \Phi_{n-1}, \tag{1.14}$$

$$\frac{\partial^2 \Phi_n}{\partial r^2} = -R'' \Phi_{n-1} + R'^2 \Phi_{n-2}, \tag{1.15}$$

$$\frac{\partial \Phi_n}{\partial t} = \frac{\eta^{n-2}}{2} \{ n \mathrm{i}^n \operatorname{erfc}[R(r)\eta^{-1}] + R\eta^{-1} \mathrm{i}^{n-1} \operatorname{erfc}[R(r)\eta^{-1}] \}. \tag{1.16}$$

But

$$n\mathrm{i}^n \operatorname{erfc} \xi + \xi \mathrm{i}^{n-1} \operatorname{erfc} \xi = \tfrac{1}{2}\mathrm{i}^{n-2} \operatorname{erfc} \xi, \tag{1.17}$$

so that (1.16) yields

$$\frac{\partial \Phi_n}{\partial t} = \tfrac{1}{4}\eta^{n-2}\mathrm{i}^{n-2}\operatorname{erfc}\left[R(r)\eta^{-1}\right] = \tfrac{1}{4}\Phi_{n-2}. \tag{1.18}$$

Now, (1.9) may be written as

$$u = T_0 + \sum_{n=0}^{\infty} a_n(r)\Phi_n \tag{1.19}$$

whence, on use of (1.14), (1.15), and (1.18), we obtain

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} a_n'\Phi_n - R' \sum_{n=0}^{\infty} a_n\Phi_{n-1}, \tag{1.20}$$

$$\frac{\partial^2 u}{\partial r^2} = \sum_{n=0}^{\infty} a_n''\Phi_n - 2R' \sum_{n=0}^{\infty} a_n'\Phi_{n-1} - R'' \sum_{n=0}^{\infty} a_n\Phi_{n-1} + R'^2 \sum_{n=0}^{\infty} a_n\Phi_{n-2}, \tag{1.21}$$

$$\frac{\partial u}{\partial t} = \tfrac{1}{4} \sum_{n=0}^{\infty} a_n\Phi_{n-2}. \tag{1.22}$$

It is noted that in the above we adopt the notation

$$\Phi_{-n} = \eta^{-n}\mathrm{i}^{-n} \operatorname{erfc}\left[R(r)\eta^{-1}\right], \tag{1.23}$$

where

$$\mathrm{i}^{-n} \operatorname{erfc} \xi = (-1)^n \, \mathrm{d}^n \operatorname{erfc} \xi / \mathrm{d}\xi^n. \tag{1.24}$$

Substitution of (1.20)–(1.22) into (1.1) now yields

$$\sum_{n=0}^{\infty} \left[\kappa\{ra_n'' + (2\nu+1)a_n'\}\Phi_n - \kappa\{2rR'a_n' + rR''a_n + (2\nu+1)R'a_n\}\Phi_{n-1} + r\{\kappa R'^2 - \tfrac{1}{4}\rho c\}a_n\Phi_{n-2}\right] = 0, \tag{1.25}$$

and the independence of the terms in Φ_n, $n = 0, 1, 2, \ldots,$ requires that

$$\kappa\{ra_n'' + (2\nu+1)a_n'\} - \kappa\{2rR'a_{n+1}' + rR''a_{n+1} + (2\nu+1)R'a_{n+1}\} + r\{\kappa R'^2 - \tfrac{1}{4}\rho c\}a_{n+2} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.26}$$

while the terms in Φ_{-1} and Φ_{-2} successively yield

$$-\kappa\{2rR'a_0' + rR''a_0 + (2\nu+1)R'a_0\} + r\{\kappa R'^2 - \tfrac{1}{4}\rho c\}a_1 = 0 \tag{1.27}$$

and

$$r\{\kappa R'^2 - \tfrac{1}{4}\rho c\} a_0 = 0. \tag{1.28}$$

Thus, if $a_0 \neq 0$, or if $a_0 = 0$ and $a_1 \neq 0$, then

$$\kappa R'^2 - \tfrac{1}{4}\rho c = 0 \tag{1.29}$$

so that $R(r) = R^{\pm}(r)$, where

$$R^{\pm}(r) = \pm \frac{1}{(4k)^{1/2}}(r-a) + R^{\pm}(a) \tag{1.30}$$

and $k = \kappa/\rho c$ is the diffusivity of the medium. In the sequel, we set $R^{\pm}(a) = 0$.

Accordingly, we obtain the representation

$$u = T_0 + \sum_{n=0}^{\infty} \{a_n^+(r)\eta^n \mathrm{i}^n \operatorname{erfc}[R^+(r)\eta^{-1}] + a_n^-(r)\eta^n \mathrm{i}^n \operatorname{erfc}[R^-(r)\eta^{-1}]\}, \tag{1.31}$$

where the $a_n^{\pm}(r)$ are determined iteratively by the differential-difference relations

$$\pm 2ra_{n+1}^{\pm\prime} \pm (2\nu+1)a_{n+1}^{\pm} - (4k)^{1/2}\{ra_n^{\pm\prime\prime} + (2\nu+1)a_n^{\pm\prime}\} = 0, \tag{1.32}$$

$$n = 0, 1, 2, \ldots,$$

$$2ra_0^{\pm\prime} + (2\nu+1)a_0^{\pm} = 0. \tag{1.33}$$

Thus,

$$\pm r^{(2\nu+1)/2}a_{n+1}^{\pm} = \pm a^{(2\nu+1)/2}\bar{a}_{n+1}^{\pm} + (k)^{1/2}\int_a^r \frac{(\sigma^{2\nu+1}a_n^{\pm\prime})'}{\sigma^{(2\nu+1)/2}}\,\mathrm{d}\sigma, \tag{1.34}$$

$$n = 0, 1, 2, \ldots,$$

$$a_0^{\pm} = \bar{a}_0^{\pm}\left(\frac{a}{r}\right)^{(2\nu+1)/2}, \quad a_0^{\pm} \neq 0, \tag{1.35}$$

where here and throughout, barred quantities denote values of $r = a$.

2. The initial condition

The expression (1.31) shows that

$$\lim_{t\to 0^+} u = T_0 + 2\sum_{n=0}^{\infty} \frac{\bar{a}_n(r-a)^n}{(4k)^{n/2}n!}, \quad r > a, \tag{2.1}$$

since

$$\lim_{t\to 0^+} t^{n/2}\mathrm{i}^n \operatorname{erfc}\left\{\frac{(r-a)}{(4kt)^{1/2}}\right\} = 0 \tag{2.2}$$

and

$$\lim_{t\to 0^+} t^{n/2}\mathrm{i}^n \operatorname{erfc}\left\{-\frac{(r-a)}{(4kt)^{1/2}}\right\} = \frac{2(r-a)^n}{(4k)^{n/2}n!}, \quad r>a,\ n=0,1,2,\dots. \tag{2.3}$$

In the half-space case $\nu = -\frac{1}{2}$ the differential-difference relations (1.26)–(1.27) for the a_n^- admit the solutions

$$a_n^- = \bar{a}_n^- = \begin{cases} \dfrac{U^{(0)}(a) - T_0}{2}, & n=0, \\ \dfrac{4(k)^{n/2}U^{(n)}(a)}{2}, & n \in \mathbb{Z}^+, \end{cases} \tag{2.4}$$

which allow the general analytic initial condition

$$\lim_{t\to 0^+} u = \sum_{n=0}^{\infty} U^{(n)}(a)\frac{(r-a)^n}{n!} = U(r) \tag{2.5}$$

to be satisfied.

If $\nu \neq -\frac{1}{2}$, the arbitrary parameters $\bar{a}_n^-$ are available in the expressions for the a_n^- for approximation to a specified initial temperature distribution $U(r)$.

3. The nonlinear boundary condition and boundedness

Since

$$u = T_0 + \sum_{n=0}^{\infty}\left\{a_n^+(r)\eta^n\mathrm{i}^n \operatorname{erfc}\left[\frac{(r-a)}{(4k)^{1/2}\eta}\right] + a_n^-(r)\eta^n\mathrm{i}^n \operatorname{erfc}\left[-\frac{(r-a)}{(4k)^{1/2}\eta}\right]\right\} \tag{3.1}$$

and

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty}\left\{a_n^{+\prime}\eta^n\mathrm{i}^n \operatorname{erfc}\left[\frac{(r-a)}{(4k)^{1/2}\eta}\right] - \frac{a_n^+}{(4k)^{1/2}}\eta^{n-1}\mathrm{i}^{n-1} \operatorname{erfc}\left[\frac{(r-a)}{(4k)^{1/2}\eta}\right]\right.$$

$$\left. + a_n^{-\prime}\eta^n\mathrm{i}^n \operatorname{erfc}\left[-\frac{(r-a)}{(4k)^{1/2}\eta}\right] + \frac{a_n^-}{(4k)^{1/2}}\eta^{n-1}\mathrm{i}^{n-1} \operatorname{erfc}\left[-\frac{(r-a)}{(4k)^{1/2}\eta}\right]\right\}, \tag{3.2}$$

it follows that

$$u(a,\,t)=T_0+\sum_{n=0}^{\infty}\left[\bar{a}_n^{+}+\bar{a}_n^{-}\right]\eta^n/2^n\Gamma\left(\tfrac{1}{2}n+1\right)\big], \tag{3.3}$$

$$\frac{\partial u}{\partial r}(a,\,t)=\sum_{n=0}^{\infty}\left\{\left[\bar{a}_n^{+\prime}+\bar{a}_n^{-\prime}\right]\eta^n/\left[2^n\Gamma\left(\tfrac{1}{2}n+1\right)\right]\right.$$

$$\left.+\frac{\left[\bar{a}_n^{-}-\bar{a}_n^{+}\right]}{(4k)^{1/2}}\eta^{n-1}\Bigg/\left[2^{n-1}\Gamma\left(\frac{n-1}{2}+1\right)\right]\right\} \tag{3.4}$$

since

$$\mathrm{i}^n\operatorname{erfc}(0)=1/\left[2^n\Gamma\left(\tfrac{1}{2}n+1\right)\right],\quad n=0,\,1,\,2,\ldots. \tag{3.5}$$

Thus, the nonlinear boundary condition (1.2) requires that

$$c_1\Phi\left[T_0+\sum_{n=0}^{\infty}\left[\bar{a}_n^{+}+\bar{a}_n^{-}\right]\eta^n/\left[2^n\Gamma\left(\tfrac{1}{2}n+1\right)\right]\right]$$

$$+c_2\sum_{n=0}^{\infty}\left\{\left[\bar{a}_n^{+\prime}+\bar{a}_n^{-\prime}\right]\eta^n/\left[2^n\Gamma\left(\tfrac{1}{2}n+1\right)\right]\right.$$

$$\left.\left.+\frac{\left[\bar{a}_n^{-}-\bar{a}_n^{+}\right]}{(4k)^{1/2}}\eta^{n-1}\Bigg/\left[2^{n-1}\Gamma\left(\frac{n-1}{2}+1\right)\right]\right\}\right]=\sum_{n=0}^{\infty}\psi_n\eta^n/n!. \tag{3.6}$$

Following the method adopted by Tao [1] in his analysis of half-space problems, the conditions imposed on the $\bar{a}_n^{\pm}$ by the relation (3.6) may be obtained by application of the result

$$D_\eta^N\left[c_1\Phi(u(a,\,t))+c_2\frac{\partial u}{\partial r}(a,\,t)\right]\Bigg|_{\eta=0}=\psi_N,\quad N=0,\,1,\,2,. \tag{3.7}$$

These conditions are set forth below:

$N=0$: In this case, we obtain

$$c_1\Phi\left[T_0+\left[\bar{a}_0^{+}+\bar{a}_0^{-}\right]\mathrm{i}^0\operatorname{erfc}0\right]+c_2\left[\bar{a}_0^{+\prime}+\bar{a}_0^{-\prime}+\frac{1}{(4k)^{1/2}}\left(\bar{a}_1^{-}-\bar{a}_1^{+}\right)\right]\mathrm{i}^0\operatorname{erfc}0=\psi_0, \tag{3.8}$$

together with, if $c_2\neq 0$,

$$\bar{a}_0^{-}=\bar{a}_0^{+}. \tag{3.9}$$

$N=1$: Since

$$D_\eta \Phi(u_a) = \Phi'(u_a) D_\eta u_a,$$

it is seen that

$$c_1 \Phi'\left[T_0 + \left[\bar{a}_0^+ + \bar{a}_0^-\right] \mathrm{i}^0 \operatorname{erfc} 0\right](\bar{a}_1^+ + \bar{a}_1^-)\mathrm{i}^1 \operatorname{erfc} 0$$

$$+c_2\left[\bar{a}_1^{+\prime} + \bar{a}_1^{-\prime} + \frac{(\bar{a}_2^- - \bar{a}_2^+)}{(4k)^{1/2}}\right]\mathrm{i}^1 \operatorname{erfc} 0 = \psi_1. \tag{3.10}$$

$N=2$: Since

$$D_\eta^2 \Phi(u_a) = \Phi''(u_a)(D_\eta u_a)^2 + \phi'(u_a) D_\eta^2 u_a,$$

it follows that

$$c_1\Big\{\Phi''\left[T_0 + \left[\bar{a}_0^+ + \bar{a}_0^-\right]\mathrm{i}^0 \operatorname{erfc} 0\right]\left((\bar{a}_1^+ + \bar{a}_1^-)\mathrm{i}^1 \operatorname{erfc} 0\right)^2$$

$$+\Phi'\left[T_0 + \left[\bar{a}_0^+ + \bar{a}_0^-\right]\mathrm{i}^0 \operatorname{erfc} 0\right] 2!(\bar{a}_2^+ + \bar{a}_2^-)\mathrm{i}^2 \operatorname{erfc} 0\Big\}$$

$$+c_2\left[\bar{a}_2^{+\prime} + \bar{a}_2^{-\prime} + \frac{\bar{a}_3^- - \bar{a}_3^+}{(4k)^{1/2}}\right] 2!\ \mathrm{i}^2 \operatorname{erfc} 0 = \psi_2. \tag{3.11}$$

N: In general, Faa de Bruno's theorem [7] shows that

$$D_\eta^N[\Phi(u(a,t))]\big|_{\eta=0} = N! \sum_{n=0}^{N} Z_n^N(u_a) \left.\frac{\partial^n \Phi}{\partial u_a^n}\right|_{\eta=0}, \quad N=1,2,\ldots, \tag{3.12}$$

where

$$Z_n^N(u_a) = \sum_{\beta_{N,n,r}} \prod_{r=1}^{N} \left(D_\eta^r u_a / r!\right)^{\beta_{N,n,r}} / \beta_{N,n,r}!, \ N=1,2,\ldots, \tag{3.13}$$

and the sum $\Sigma_{\beta_{N,n,r}}$ is extended over all multinomial coefficients, that is, over non-negative integers $\beta_{N,n,r}$ such that

$$\sum_{r=1}^{N} \beta_{N,n,r} = n, \quad \sum_{r=1}^{N} r\beta_{N,n,r} = N. \tag{3.14,3.15}$$

Since

$$D_\eta^r u_a\big|_{\eta=0} = (\bar{a}_r^+ + \bar{a}_r^-) r!\ \mathrm{i}^r \operatorname{erfc} 0, \quad r=1,2,\ldots, \tag{3.16}$$

and

$$D_\eta^N \frac{\partial u}{\partial r}(a, t)\bigg|_{\eta=0} = \left[\bar{a}_N^{+\prime} + \bar{a}_N^{-\prime} + \frac{(\bar{a}_{N+1}^- - \bar{a}_{N+1}^+)}{(4k)^{1/2}}\right] N!\, \mathrm{i}^N \operatorname{erfc} 0, \tag{3.17}$$

$N = 0, 1, 2, \ldots,$

the condition (3.7) yields

$$c_1 \sum_{n=0}^{N} Z_n^N(u_a) \frac{\partial^n \Phi}{\partial u_a^n}\bigg|_{\eta=0} + c_2 \left[\bar{a}_N^{+\prime} + \bar{a}_N^{-\prime} + \frac{(\bar{a}_{N+1}^- - \bar{a}_{N+1}^+)}{(4k)^{1/2}}\right] \mathrm{i}^N \operatorname{erfc} 0 = \frac{\psi_N}{N!}, \tag{3.18}$$

$N = 1, 2, \ldots,$

where

$$Z_n^N(u_a) = \sum_{\beta_{N,n,r}} \prod_{r=1}^{N} \left[(\bar{a}_r^+ + \bar{a}_r^-)\mathrm{i}^r \operatorname{erfc} 0\right]^{\beta_{N,n,r}} / \beta_{N,n,r}!. \tag{3.19}$$

It is recalled that the $\bar{a}_N^-$ are determined by the initial condition (1.3). In what follows, we proceed with the initial condition that requires

$$\lim_{t \to 0^+} u = T_0, \tag{3.20}$$

so that $\bar{a}_n(r) = \bar{a}_n^- = 0$, $n = 0, 1, 2, \ldots,$ and the relations (3.8)–(3.11) yield

$$\bar{a}_0^+ = \bar{a}_0^- = 0, \tag{3.21}$$

$$c_1 \Phi(T_0) - \frac{c_2 \bar{a}_1^+}{(4k)^{1/2}} = \psi_0, \tag{3.22}$$

$$\frac{1}{2\Gamma(\frac{3}{2})}\left[c_1 \Phi'(T_0)\bar{a}_1^+ + c_2\left\{\bar{a}_1^{+\prime} - \frac{\bar{a}_2^+}{(4k)^{1/2}}\right\}\right] = \psi_1 \tag{3.23}$$

and

$$c_1\left[\Phi''(T_0)\left(\frac{\bar{a}_1^+}{2\Gamma(\frac{3}{2})}\right)^2 + \tfrac{1}{2}\phi'(T_0)\bar{a}_2^+\right] + \frac{c_2}{2}\left[\bar{a}_2^{+\prime} - \frac{\bar{a}_3^+}{(4k)^{1/2}}\right] = \psi_2. \tag{3.24}$$

In general, (3.18) and (3.19) show that

$$c_1 \sum_{n=0}^{N} Z_n^N(T_0) \frac{\partial^n \Phi}{\partial u_a^n}\bigg|_{\eta=0} + \frac{c_2}{2^N \Gamma(\frac{1}{2}N + 1)}\left[\bar{a}_N^{+\prime} - \frac{\bar{a}_{N+1}^+}{(4k)^{1/2}}\right] = \psi_N \tag{3.25}$$

where

$$Z_n^N(T_0) = \sum_{\beta_{N,n,r}} \prod_{n=1}^{N} \left[\bar{a}_r^+/2^r\Gamma\left(\frac{r}{2}+1\right)\right]^{\beta_{N,n,r}} / \beta_{N,n,r}!. \tag{3.26}$$

The recurrence relation (1.34) with $n = 0$ now shows that

$$a_1^+ = \bar{a}_1^+\left(\frac{a}{r}\right)^{(2\nu+1)/2}, \tag{3.27}$$

where, from (3.22),

$$\bar{a}_1^+ = \frac{(4k)^{1/2}}{c_2}\left[c_1\Phi(T_0) - \psi_0\right], \quad c_2 \neq 0. \tag{3.28}$$

Thus, from (1.34) with $n = 1$,

$$a_2^+ = \left(\frac{a}{r}\right)^{(2\nu+1)/2}\left[\bar{a}_2^+ + (k)^{1/2}\frac{(2\nu+1)(2\nu-1)}{4}\left[\frac{1}{r} - \frac{1}{a}\right]\bar{a}_1^+\right] \tag{3.29}$$

where (3.23) shows that

$$\bar{a}_2^+ = (4k)^{1/2}\left[\frac{c_1}{c_2}\Phi'(T_0)\bar{a}_1^+ - \frac{2}{c_2}\Gamma\left(\tfrac{3}{2}\right)\psi_1 - \left(\frac{2\nu+1}{2}\right)\frac{\bar{a}_1^+}{a}\right], \quad c_2 \neq 0. \tag{3.30}$$

In general, the a_n^+ may now be readily generated by combination of the recurrence relations (1.34)–(1.35) and the relations (3.25)–(3.26) which determine the constants of integration $\bar{a}_n^+$.

To complete this section it is noted that, since $a_n^-(r) = 0$ in the present discussion,

$$\lim_{r\to\infty} u = T_0 + \lim_{r\to\infty}\left[\sum_{n=0}^{\infty} a_n^+\eta^n \mathrm{i}^n \operatorname{erfc}\left[\frac{(r-a)}{(4k)^{1/2}\eta}\right]\right]$$

$$= T_0, \tag{3.31}$$

and the boundedness condition (1.4) is met.

4. The Stefan-Boltzmann radiation condition

In this section we consider in detail the results of the last section as they apply to a body at absolute temperature $u(r, t)$ subject to black-body radiation into a medium of absolute temperature T_e. The boundary condition for this case is

$$\frac{\partial u}{\partial r}(a, t) = \frac{\sigma E}{\kappa}\left[u^4(a, t) - T_e^4\right], \tag{4.1}$$

where σ is the Stefan-Boltzmann constant and E is the emissivity of the surface. We also assume that the initial temperature is constant and equal to T_0. It will be convenient for our numerical results obtained in the next section to recast our boundary-value problem in terms of dimensionless variables.

We introduce the dimensionless variables

$$\hat{u} = u/T_1, \quad \hat{r} = r/a, \quad \hat{t} = kt/a^2,$$
$$\hat{T}_e = T_e/T_1, \quad \hat{T}_0 = T_0/T_1, \quad \alpha = \frac{\sigma E a T_1^3}{\kappa}, \tag{4.2}$$

where the reference temperature T_1 is taken to be the initial temperature T_0 if $T_0 \neq 0$, (and hence $\hat{T}_0 = 1$). If $T_0 = 0$, we take $T_1 = T_e$, (so that $\hat{T}_e = 1$). Our dimensionless scheme then includes the case of non-zero initial temperature and zero initial temperature. The trivial case in which initial and ambient temperature are both zero is excluded. If we use the dimensionless variables (4.2) but omit the carets, the nonlinear boundary-value problem becomes

$$\frac{\partial}{\partial r}\left(r^{2\nu+1}\frac{\partial u}{\partial r}\right) = r^{2\nu+1}\frac{\partial u}{\partial t}, \qquad r > 1,\ t > 0; \tag{4.3}$$

$$u(r, 0+) = T_0, \qquad r > 1; \tag{4.4}$$

$$\frac{\partial u}{\partial r}(1, t) = \alpha\left[u^4(1, t) - T_e^4\right], \quad t > 0. \tag{4.5}$$

The solution to this problem can now be written down by employing the results of the previous section with $c_1 = -\alpha$, $c_2 = 1$ together with

$$\psi_n = \begin{cases} -\alpha T_e^4, & n = 0, \\ 0, & n = 1, 2, \ldots, \end{cases}$$

and $\Phi(u) = u^4$. In addition we must set $a = 1$, $k = 1$ to take into account the non-dimensional scheme. We find that

$$u(r, t) = T_0 + \sum_{n=1}^{\infty} a_n(r)\eta^n \mathrm{i}^n \operatorname{erfc}(R\eta^{-1}), \quad \eta = \sqrt{t}, \tag{4.7}$$

where

$$R = \frac{r-1}{2}, \tag{4.8}$$

and the a_n satisfy

$$2ra'_{n+1} + (2\nu+1)a_{n+1} - 2\left[ra''_n + (2\nu+1)a'_n\right] = 0, \quad n = 1, 2, \ldots, \tag{4.9}$$

$$a_1(r) = \bar{a}_1\left(\frac{1}{r}\right)^{(2\nu+1)/2}. \tag{4.10}$$

In obtaining this last result we have used $a_n^- = 0$ and defined $a_n \equiv a_n^+$. Equation (4.9) is analogous to the transport equation of ray-series methods employed in wave-propagation problems [3], [6]. Its solution is readily found to be

$$a_n(r) = \left(\frac{1}{r}\right)^{(2\nu+1)/2}\left\{\bar{a}_n + \sum_{j=2}^{n} a_{nj}(r^{1-j}-1)\right\}, \quad n=2,3,\dots,\ \bar{a}_n \equiv a_n(1), \tag{4.11}$$

where the a_{nj} satisfy the recurrence relations

$$a_{n+1,j+1} = \begin{cases} \dfrac{(2\nu-1)(2\nu+1)}{4}\bar{a}_1, & j=1,\ n=1, \\ \dfrac{(2\nu-1)(2\nu+1)}{4}\left[\bar{a}_n - \displaystyle\sum_{i=2}^{n} a_{ni}\right], & j=1,\ n\geqslant 2, \\ \dfrac{(2\nu-2j+1)(2\nu+2j-1)}{4j}a_{nj}, & 2\leqslant j\leqslant n,\ n\geqslant 2. \end{cases} \tag{4.12}$$

The coefficients in (4.7) can now be completely determined through (4.11) and (4.12) provided we know $\bar{a}_n$. This quantity is computed using the boundary condition on $r=1$. Employing $\Phi(u)=u^4$ and (4.6) in (3.25) yields

$$\bar{a}_1 = -2\alpha(T_0^4 - T_e^4), \tag{4.13}$$

$$\bar{a}_{n+1} = 2\bar{a}_n' - 2^{n+1}\Gamma(1+\tfrac{1}{2}n)\alpha\left[\sum_{i=0}^{n}\left(\sum_{k=0}^{n-i} A_{n-i-k+1}A_{k+1}\right)\left(\sum_{k=0}^{i} A_{i-k+1}A_{k+1}\right)\right], \tag{4.14}$$

$$n=1,2,\dots,\ \bar{a}_n' \equiv a_n'(1),$$

where

$$A_1 = T_0,$$
$$A_n = \bar{a}_{n-1}/2^{n-1}\Gamma\left(1+\frac{n-1}{2}\right), \quad n=2,3,\dots. \tag{4.15}$$

We may now compute as many terms as we wish in the series (4.7) by employing the sequence of formulae (4.11)–(4.15). At each step, a_{n+1} is computed from a_n by finding $a_{n+1,j+1}$, $(1\leq j\leq n)$, using (4.12) and by finding $\bar{a}_{n+1}$ from (4.14), where $\bar{a}_n'$ in (4.14) is found by using

$$\bar{a}_1' = -\left(\frac{2\nu+1}{2}\right)\bar{a}_1,$$

$$\bar{a}_n' = -\left(\frac{2\nu+1}{2}\right)\bar{a}_n + \sum_{j=2}^{n} a_{nj}(1-j), \quad n\geqslant 2. \tag{4.16}$$

To illustrate the use of the above formulae, let us calculate explicitly the first two terms in the series for $u(r,t)$. From (4.10) and (4.13)

$$a_1(r) = 2\alpha(T_e^4 - T_0^4)\left(\frac{1}{r}\right)^{(2\nu+1)/2}. \tag{4.17}$$

Setting $n=1$ in (4.14) gives

$$\bar{a}_2 = 2\bar{a}_1' - 4\alpha\Gamma(\tfrac{3}{2})\sum_{i=0}^{1}\left(\sum_{k=0}^{1-i} A_{2-i-k}A_{k+1}\right)\left(\sum_{k=0}^{i} A_{i-k+1}A_{k+1}\right)$$

$$= 2\bar{a}_1' - 16\alpha\Gamma(\tfrac{3}{2})A_1^3 A_2. \tag{4.18}$$

Employing (4.16) to find $\bar{a}_1'$ and (4.15) to obtain A_1, A_2, (4.18) yields

$$\bar{a}_2 = -2\left[\frac{2\nu+1}{2} + 4\alpha T_0^3\right]\bar{a}_1. \tag{4.19}$$

Now

$$a_2(r) = \left(\frac{1}{r}\right)^{(2\nu+1)/2}\left\{\bar{a}_2 + a_{22}\left(\frac{1}{r}-1\right)\right\}$$

where, from (4.12),

$$a_{22} = \frac{(2\nu-1)(2\nu+1)}{4}\bar{a}_1.$$

Thus,

$$a_2(r) = \left(\frac{1}{r}\right)^{(2\nu+1)/2}\left[-2\left(\frac{2\nu+1}{2} + 4\alpha T_0^3\right) + \frac{(2\nu-1)(2\nu+1)}{4}\left(\frac{1}{r}-1\right)\right]$$

$$\times 2\alpha\left(T_e^4 - T_0^4\right). \tag{4.20}$$

so that (4.17), (4.20), in (4.7) yield the small-time approximation

$$u \approx T_0 + a_1(r)\eta \text{ i}^1 \operatorname{erfc}\left(\frac{r-1}{2\eta}\right) + a_2(r)\eta^2 \text{ i}^2 \operatorname{erfc}\left(\frac{r-1}{2\eta}\right). \tag{4.21}$$

To proceed to calculate more terms by hand is unwieldy. Nevertheless, the recurrence formulae (4.12) and (4.14) are in a form suitable for coding in a programming language. We shall explore this idea in the next section. In particular we shall use the Bergman-type series solution to obtain numerical results for the Stefan-Boltzmann problem.

5. Numerical results and Padé extension of the Bergman series

Numerical results for the Stefan-Boltzmann problem are obtained from the formal series solution (4.7). We begin by programming the sequence of recurrence formulae (4.12)–(4.16) constructed in the previous section. We may thus sum as many terms in the series (4.7) as we wish. In view of the complicated nature of the recurrence formulae it is difficult to discover the region of convergence of the formal series solution. For practical purposes this is not necessary. By using an increasing number of terms we can estimate the range of the independent variables for which the series solution is convergent.

The quantity

$$\alpha_n \equiv 2^n \Gamma(1+\tfrac{1}{2}n) = 1/\mathrm{i}^n \operatorname{erfc}(0), \tag{5.1}$$

which appears in (4.14), grows rapidly with n. It is therefore necessary to scale out α_n in order to numerically implement our recurrence formulae. To this end let

$$a_n(r) = b_n(r)\alpha_n, \tag{5.2}$$

so that

$$u(r, t) = T_0 + \sum_{n=1}^{\infty} b_n(r)\eta^n f_n(R\eta^{-1}), \tag{5.3}$$

where

$$f_n(z) = \alpha_n \mathrm{i}^n \operatorname{erfc}(z). \tag{5.4}$$

The functions $f_n(z)$ satisfy the recurrence formula

$$f_n(z) = -\frac{z}{n\gamma_{n-1}} f_{n-1}(z) + f_{n-2}(z), \tag{5.5}$$

where

$$\gamma_n = \frac{\alpha_n}{\alpha_{n+1}}. \tag{5.6}$$

The sequence γ_n can be calculated using

$$\gamma_n \gamma_{n-1} = \frac{1}{2(n+1)}, \quad n = 1, 2, \ldots, \gamma_0 = \frac{1}{\sqrt{\pi}}. \tag{5.7}$$

To compute the functions $b_n(r)$ we employ the following formulae obtained by using (5.2) in (4.10)–(4.16):

$$b_1(r) = \bar{b}_1 \left(\frac{1}{r}\right)^{(2\nu+1)/2}, \tag{5.8}$$

$$b_n(r) = \left(\frac{1}{r}\right)^{(2\nu+1)/2} \left\{ \bar{b}_n + \sum_{j=2}^{n} b_{nj}(r^{1-j} - 1) \right\}, \quad n \geqslant 2, \tag{5.9}$$

$$\bar{b}_n \equiv b_n(1) \tag{5.10}$$

$$b_{n+1,j+1}=\begin{cases}\gamma_1\dfrac{(2\nu-1)(2\nu+1)}{4}\bar{b}_1, & j=1,\ n=1,\\ \gamma_n\dfrac{(2\nu-1)(2\nu+1)}{4}\left[\bar{b}_n-\sum\limits_{i=2}^{n}b_{ni}\right], & j=1,\ n\geqslant 2,\\ \gamma_n\dfrac{(2\nu-2j+1)(2\nu+2j-1)}{4j}b_{nj}, & 2\leqslant j\leqslant n,\ n\geqslant 2,\end{cases} \tag{5.11}$$

$$\bar{b}_1=-\frac{2\alpha\left(T_0^4-T_e^4\right)}{\sqrt{\pi}}, \tag{5.12}$$

$$\bar{b}_{n+1}=\gamma_n\left\{2\bar{b}_n^1-2\alpha\sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}\bar{b}_{n-i-k}\bar{b}_k\right)\left(\sum_{k=0}^{i}\bar{b}_{i-k}\bar{b}_k\right)\right\},$$

$$n=1,2,\ldots,\ \bar{b}_0\equiv T_0, \tag{5.13}$$

$$\bar{b}_1^1=-\left(\frac{2\nu+1}{2}\right)\bar{b}_1, \tag{5.14}$$

$$\bar{b}_n^1=-\left(\frac{2\nu+1}{2}\right)\bar{b}_n+\sum_{j=2}^{n}b_{nj}(1-j),\quad n=2,3,\ldots. \tag{5.15}$$

We concentrate on obtaining results for the surface temperature $u(1,\,t)$ since this is most frequently the quantity of physical interest. Since

$$f_n(0)=1,$$

we have from (5.3):

$$u(1,\,t)=\sum_{n=0}^{\infty}\bar{b}_n\eta^n,\quad \bar{b}_0\equiv T_0. \tag{5.16}$$

Summing this series for various choices of the dimensionless parameters reveals that it is convergent only for small values of η. To obtain an accurate global approximation to the surface temperature we now employ Padé approximants.

By definition, [9],

$$[L/M]=P_L(\eta)/Q_M(\eta) \tag{5.17}$$

is the L, M Padé approximant to $u(1,\,t)$ where P_L, Q_M are polynomials of degree at most L, M respectively and their coefficients are determined by

$$\frac{P_L(\eta)}{Q_M(\eta)}=\sum_{n=0}^{L+M}\bar{b}_n\eta^n+\mathrm{O}\left(\eta^{L+M+1}\right). \tag{5.18}$$

If we let

$$P_L(\eta)=\sum_{n=1}^{L+1}p_n\eta^{n-1},\quad Q_M(\eta)=\sum_{n=1}^{M+1}q_n\eta^{n-1}, \tag{5.19}$$

then from (5.18) we can easily construct a system of linear equations for the coefficients of $P_L(\eta)$ and $Q_M(\eta)$.

In order to obtain a good approximation for $u(1, t)$ we should build in any information available about the surface temperature. In particular, we know that

$$\lim_{t\to\infty} u(1, t) = T_e. \tag{5.20}$$

To obtain this asymptotic behavior we choose

$$L = M, \; q_{L+1} = 1, \quad p_{L+1} = T_e. \tag{5.21}$$

The coefficients in the resulting rational approximation are then found to satisfy

$$\sum_{j=1}^{L} \bar{b}_{i+L-j} q_j = T_e \delta_{L+1,i+1} - a_i, \quad i = 1, \ldots, L, \tag{5.22}$$

$$p_i = \sum_{j=1}^{i} \bar{b}_{i-j} q_j, \quad i = 1, \ldots, L. \tag{5.23}$$

We denote by $F_{L/L}(\eta)$ the approximant which uses the coefficients obtained by solving (5.22) and (5.23). Typically L will be no greater than 15, so no effort at algorithm efficiency is required. We simply employ a standard linear-equation solver and then evaluate the corresponding rational function. As an aid in selecting L, we use the fact that $u(1, t)$ is neither zero nor singular for $t > 0$. Thus, in choosing an appropriate value of L we rule out any rational approximation $F_{L/L}(\eta)$ whose numerator or denominator has a zero in $\eta > 0$.

Numerical results are obtained for $T_0 = 0$ (so $T_e = 1$) and for $T_0 = 1$, $T_e = 0.5$ and are displayed graphically in Figs. 1 and 2. We consider the cases $\nu = -\frac{1}{2}, 0, \frac{1}{2}$, where $\nu = -\frac{1}{2}$ corresponds to a half-space, while $\nu = 0, \frac{1}{2}$ correspond respectively to regions bounded internally by a cylinder and a sphere. All results are for $\alpha = 1$. When $\nu = -\frac{1}{2}$ there is no

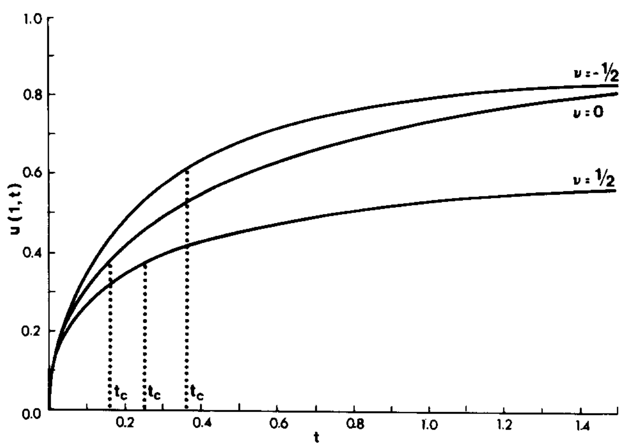

Figure 1. Variation of surface temperature with time for $T_0 = 0$, $T_e = 1$, $\alpha = 1$.

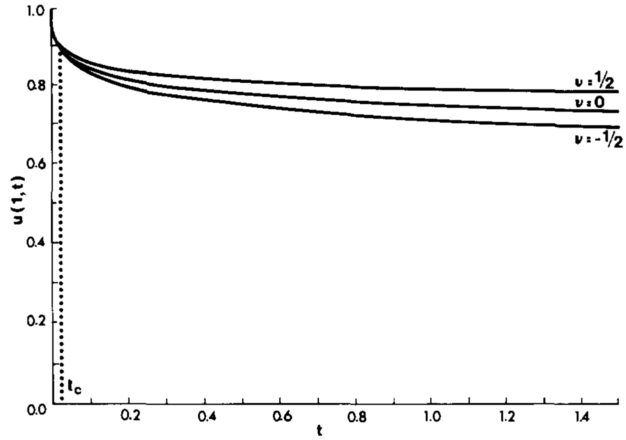

Figure 2. Variation of surface temperature with time for $T_0 = 1$, $T_p = 0.5$, $\alpha = 1$.

natural reference length and this choice of α corresponds to choosing $\kappa/e\sigma T_1^3$ as the reference length.

By summing an increasing number of terms in the series (5.16) we can estimate the interval of convergence of this series. The quantity t_c appearing in Figs. 1 and 2 is an estimate of the non-dimensional time beyond which (5.16) is divergent.

6. Concluding remarks

It is a straightforward procedure to construct Bergman-type series solutions for non-linear boundary-value problems associated with the heat equation. As demonstrated here, numerical results for the surface temperature obtained by summing such series are valid only for small time. This paper shows that numerical results can be obtained for much greater times by employing Padé approximants. In fact good global approximations are obtained for the surface temperature by building in the asymptotic limit and by sensibly choosing the degree L of the rational approximation $F_{L/L}$. We picked L to be less than 15 and from that range chose rational functions which are neither singular nor zero. For each set of results obtained, two approximants from this admissible set were employed. Results obtained from such pairs of rational functions were found to agree at least to three figures and for some sets of results to six figures throughout the entire domain. This fact gives confidence in our numerical results. Also, the case shown in Fig. 1 for $\nu = -\frac{1}{2}$ has been considered in [10] and results found there are graphically indistinguishable from ours. Since the construction of Bergman-type series is a direct procedure which involves only simple operations, this method allied with Padé approximants provides an important new means of obtaining numerical results from non-linear boundary-value problems associated with the heat equation.

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